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LETTER TO THE EDITOR

On macroscopic energy gap for q-quantum mechanical systems

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Dedicated to our daughter Nela at the occasion of her first birthday

Abstract. The q-deformed harmonic oscillator within the framework of the recently introduced Schwenk–Wess q-Heisenberg algebra is considered. It is shown, that for 'physical' values $q \sim 1$, the gap between the energy levels decreases with growing energy. Comparing with the other (real) q-deformations of the harmonic oscillator, where the gap instead increases, indicates that the formation of the macroscopic energy gap in the Schwenk–Wess q-quantum mechanics may be avoided.

The harmonic oscillator is the typical physical system which the various q-deformations are tested on [1–4]. In this letter we shall concentrate on one particular aspect of the behaviour of the spectra of the q-deformed oscillators. Namely, the gap between the energy levels increases with growing energy, in the models considered in [1–4]. As already mentioned in [4], one eventually ends up with the macroscopic gap for the high energy levels, unless the undeformed case $q=1$ is considered. Moreover, the gap between the energy levels grows exponentially with the energy if $q \neq 1$. It is questionable whether such behaviour could remain 'unseen' in our physical world if q were not equal to one.

We shall show that the q-harmonic oscillator within the framework of the recently proposed Schwenk–Wess q-deformation of the Heisenberg algebra has the gap between the energy level *decreasing* with growing energy. We shall do it within the framework of the perturbation theory, where the q-deformed Hamiltonian differs from the undeformed one by the perturbation. This approach should give reliable results since, anyway, the physics requires that $q \sim 1$.

The Schwenk–Wess q-deformation of the Heisenberg algebra [5] was motivated by the differential calculus on quantum planes [6] and it reads

$$\xi\pi - q^{-1}\pi\xi = \mathrm{i}\eta \tag{1}$$

$$\eta\pi = q\pi\eta \qquad \eta\xi = q^{-1}\xi\eta \tag{2a, b}$$

$$\eta\eta^{+} = \eta^{+}\eta = q^{-1} \tag{3}$$

where ξ and π are self-adjoint, q real and $q \geqslant 1$. The explicit representations of this algebra were also found in [5]. However, those representations were not suited for direct limiting $q \to 1$. We give another representation of this algebra, given by

$$\pi = p\frac{[\mathrm{i}xp]_q}{\mathrm{i}xp} \qquad \xi = x \qquad \eta = q^{-\mathrm{i}px} \tag{4}$$

where x and p are the standard quantum mechanical operators of the position and of the momentum respectively, obeying the commutation relations of the undeformed Heisenberg algebra, and the symbol $[\,.\,]_q$, as usual, means

$$[A]_q \equiv \frac{q^A - q^{-A}}{q - q^{-1}}. \tag{5}$$

Note that in x-representation the deformed operator π is proportional to nothing but the well known q-derivative, i.e.

$$\pi\psi(x) = -\mathrm{i}D_q\psi(x) \equiv -\mathrm{i}\,\frac{\psi(qx) - \psi(q^{-1}x)}{(q - q^{-1})x}. \tag{6}$$

Using the commutation relations of the undeformed Heisenberg algebra, we find that the operator π is self-adjoint†, as it should be. We do not say, however, that the representation (4) and the representations given by Schwenk and Wess [5] are equivalent. In fact, we have no reason to assert that an analogue of the von Neumann theorem should be valid in the q-deformed case. In the case of the standard (i.e. undeformed) quantum systems with the infinite number of degrees of freedom, however, not only the algebra of observables but also its representation have to be chosen, in order to specify the system [7]. Hence, we shall adopt the same point of view also in the present case and study the algebra (1)-(3) with its representation (4).

We pick up the Hamiltonian of the q-harmonic oscillator as follows

$$H = (\pi^2 + \omega^2\xi^2)/2 \tag{7}$$

where ω is a real parameter. We wish to identify the spectrum of (7). The standard combinations $(\pi \pm \mathrm{i}\omega\xi)/\sqrt{2\omega}$ do not play the role of the raising and lowering operators, however. The method [4] for constructing the raising and lowering operators for the q-deformed Schrödinger problems does not seem to work either, in our case. We may, however, perform the perturbation expansion in the parameter of deformation and look after the modifications of the spectrum. The eventual result will certainly give us the relevant information about the behaviour of the energy gap, because, anyway, the physical q cannot differ much from 1. Set

$$q = \mathrm{e}^{\beta} \tag{8}$$

and expand

$$\begin{aligned} H = H_0 + H_1\beta^2 + H_2\beta^4 + \ldots &\equiv \tfrac{1}{2}(p^2 + \omega^2x^2) + \tfrac{1}{2}[(\mathrm{i}px)^2p^2 + p^2(\mathrm{i}xp)^2 - 2p^2]\frac{1}{3!}\beta^2 \\ &+ \frac{1}{2}\left[\frac{(\mathrm{i}px)^4}{5!}p^2 + p^2\frac{(\mathrm{i}xp)^4}{5!} + \frac{(\mathrm{i}px)^2p^2(\mathrm{i}xp)^2}{3!3!}\right. \\ &\left. - \frac{(\mathrm{i}px)^2p^2 + p^2(\mathrm{i}xp)^2}{3\times 3!} + \frac{p^2}{3\times 5}\right]\beta^4 + \ldots. \end{aligned} \tag{9}$$

The eigenvalues $E_n(\beta^2)$ and the eigenvectors $|\psi_n(\beta^2)\rangle$ will have the following expansions

$$E_n = E_{n,0} + E_{n,1}\beta^2 + E_{n,2}\beta^4 + \ldots \tag{10}$$

$$|\psi_n\rangle = |\psi_{n,0}\rangle + |\psi_{n,1}\rangle\beta^2 + |\psi_{n,2}\rangle\beta^4 + \ldots. \tag{11}$$

† Rigorously speaking, we did not study the domains of definition of the operators π and η, given by equation (4). Such investigation would be beyond the scope of this paper.

Expanding the equation

$$(H - E_n)|\psi_n\rangle = 0 \tag{12}$$

gives

$$(H_0 - E_{n,0})|\psi_{n,0}\rangle = 0 \tag{13a}$$

$$(H_1 - E_{n,1})|\psi_{n,0}\rangle + (H_0 - E_{n,0})|\psi_{n,1}\rangle = 0 \tag{13b}$$

$$(H_2 - E_{n,2})|\psi_{n,0}\rangle + (H_1 - E_{n,1})|\psi_{n,1}\rangle + (H_0 - E_{n,0})|\psi_{n,2}\rangle = 0. \tag{13c}$$

It is easy to conclude, that

$$E_{n,1} = \langle\psi_{n,0}|H_1|\psi_{n,0}\rangle \tag{14a}$$

$$E_{n,2} = \langle\psi_{n,0}|H_2|\psi_{n,0}\rangle + \sum_{l\neq n} \frac{|\langle\psi_{l,0}|H_1|\psi_{n,0}\rangle|^2}{E_{n,0} - E_{l,0}}. \tag{14b}$$

Note, that the first term in the right-hand side of (14*b*) is usually omitted in the standard textbooks on quantum mechanics since the expansion like (9) is usually written as the sum of only two terms, i.e. the unperturbed Hamiltonian and the perturbation. Hence, e.g. the usual conclusion about the negativeness of the second correction to the vacuum energy need not be valid in our case.

The most convenient representation for the evaluation of the required matrix elements is the energy representation of the unperturbed oscillator, i.e. the representation in which the operators a, a^+, given by

$$a = \frac{p - \mathrm{i}\omega x}{\sqrt{2\omega}} \qquad a^+ = \frac{p + \mathrm{i}\omega x}{\sqrt{2\omega}} \tag{15}$$

act as follows

$$a|\psi_{n,0}\rangle = \sqrt{n}|\psi_{n-1,0}\rangle \qquad a^+|\psi_{n,0}\rangle = \sqrt{n+1}|\psi_{n+1,0}\rangle \tag{16}$$

where $n = 0, 1, \ldots$. Inserting the expressions

$$p = \frac{\sqrt{\omega}(a + a^+)}{\sqrt{2}} \qquad x = \frac{(a^+ - a)}{\mathrm{i}\sqrt{2\omega}} \tag{17}$$

into (9) we may write, e.g.

$$E_{n,1} = \frac{\omega}{2^4 \times 3!} \langle\psi_{n,0}|[(a^{+^2} - a^2 + 1)^2(a + a^+)^2 + (a + a^+)^2(a^{+^2} - a^2 - 1)^2 - 8(a + a^+)^2]|\psi_{n,0}\rangle. \tag{18}$$

Using (16), we have

$$E_{n,1} = -\frac{\omega}{2 \times 3!} \left\{ (n + \tfrac{1}{2})^3 + \frac{17}{2^2}(n + \tfrac{1}{2}) \right\}. \tag{19}$$

We observe, that the first order corrections to the energy levels are negative and increasing with n in the absolute value. Moreover, the gap between the energy levels tend to diminish with growing n. The calculation of the second-order corrections require the knowledge of the off-diagonal matrix elements of H_1 as well as of the diagonal elements of H_2. A short look at equation (9) suggests that the calculation be somewhat lengthy. Nevertheless, the careful (and several times checked) evaluation gives the following result for the energy levels up to the second order

$$E_n = \omega(n+\tfrac{1}{2})\left\{1-\beta^2\frac{1}{2\times 3!}\left[(n+\tfrac{1}{2})^2+\frac{17}{2^2}\right]\right.$$
$$\left.-\beta^4\frac{7}{2^3\times 5!}\left[(n+\tfrac{1}{2})^4+\frac{5^2\times 13}{2\times 3\times 7}(n+\tfrac{1}{2})^2-\frac{1031}{2^4\times 3\times 7}\right]+\ldots\right\}. \qquad (20)$$

With the exception of the vacuum, the second correction is also always negative and for large n the second correction to the energy levels gap decreases with growing n.

Needless to say, the result is not very helpful in seeking the exact solution. One requires the better understanding of the algebraic structure of the Schwenk–Wess algebra, which would provide a construction of the appropriate raising and lowering operators. We feel, however, that the perturbative result *is* relevant and, more generally, that the phenomenology of the Schwenk–Wess q-quantum mechanics may be of interest.

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